Large deformations of elastic bodies are typical in cases where tight packing is necessary due to particular constraints [1] or external pressure [2,3]. In many cases the systems of interest are nanoscale, e.g., DNA molecules in bacteriophage capsids [4], empty virus shells under osmotic pressure [3], and crystals of deformable, soft particles [5] such as fullerenes and carbon nanotubes [6]. There are also systems on a micrometer length scale characterized by tight packing and pronounced deformation, such as epithelial tissues [7]. The elasticity of the systems of interest is often studied in the linear elasticity regime (see, e.g., Ref. [8]) where the characteristic deformations are small. However, to properly account for energies of tightly packed and/or constrained and strongly deformed structures, one needs to consider the elastic energies in the nonlinear regime. Yet the functional relations between force (energy) and deformation are derived mostly in the small-deformation regime [9]. Although strongly deformed bodies may be studied numerically using different variants of the finite element technique, functional relations that transparently relate energy and deformation are certainly of special use. Such analytical expressions are of importance in many applications, especially in nanoscale physics and cellular biophysics [7].

In this Brief Report we introduce a spring made by rolling a piece of thin elastic sheet to form a cylindrical tube. We predict small and large deformations of such a spring situated in between two parallel plates by using an approximate but completely adequate variant of the theory of elasticity. Our work presents a way to obtain reliable approximate but completely adequate variant of the theory of elasticity. The system that we have chosen is meant to model cases of large deformations in materials that are made of thin sheets. This is motivated by thin-sheet structures such as carbon nanotubes, fullerenes, and other similar structures made of graphene sheets [10,11], cellular membranes and vesicles [7,12] (made of a lipid bilayer), and protein shells (made of protein sheets), such as virus capsids [3,13,14] and microtubules [15]. One may wonder whether our (macroscopic) experimental setup and the classical theory of elasticity can be used as a model and an approach relevant for nanoscale systems. It has been shown that the energies of nanoscale thin-shelled systems (larger than about 2 nm [11]) can be very accurately determined using the theory of elasticity [10,11] (see also Ref. [16]), thus our work directly reflects on the systems of present interest.

The tube is pressed between two parallel plates as depicted in Fig. 1. The length of the parallel plates is larger than the length of the tube \( h \), which makes the problem effectively one dimensional. As the tube deforms inextensionally, the stretching of the sheet can be neglected [17] and the elastic energy of the pressed tube is \( E_d = \kappa \int_0^L K^2 dS/2 \), where \( \kappa \) is the bending (or flexural) rigidity of the sheet and \( K = R_1^{-1} + R_2^{-1} \), where \( R_1 \) and \( R_2 \) are the principal radii of curvature at some point. We have implicitly assumed that both radii are much larger than the thickness of the sheet \( d \). If the material the sheet is made of is isotropic, the bending rigidity \( \kappa \) is related to the Young modulus and Poisson ratio \( \nu \) of the material as [17]

\[
\kappa = \frac{Ed^3}{12(1-\nu^2)}. \tag{1}
\]

Since the sheet is bent only along one direction, we can write the elastic energy in terms of the one-dimensional integral

\[E_d = k h \int_C K^2 dl/2,\]

where \( C \) is the curve outlining the shape of the cylinder base and \( dl \) is the infinitesimal arc element...
of the curve $C$ ($dS = h \, dl$). The force exerted by the pressed cylinder onto the plates can be measured by using a simple scale located below the bottom plate (see Fig. 1). It is useful to recognize that the effective mass $m_e$ measured by the scale can be thought of as a mass that presses the spring from above due to gravity (the mass of the spring being neglected here). With this picture in mind, the total energy of the system is

$$E = E_p + E_{el} = 2m_e gb + \frac{\kappa h}{2} \int_C K^2 \, dl, \quad (2)$$

where $2b$ is the separation between the plates. For a given mass $m_e$, there is an equilibrium value of $b$ at which $\partial E / \partial b = 0$. The difficult part of the problem is to find the curve $C$ that minimizes the elastic energy $E_{el}$ for a given separation between the plates $2b$; the minimization should be performed with two constraints: The length of the curve is fixed (the sheet is inextensible) and the height of the object depicted by $C$ is $2b$.

FIG. 1. (Color online) Illustration of the experimental setup.

To explore the energetics of the problem analytically, we use two qualitatively different Ritz trial solutions (Ansätze) for curve $C$: (i) the stadium-shaped curve made of two identical semicircles connected by two straight lines that touch the press plates (this profile is expected to be a good model for sufficiently large pressing forces and is often used for vesicles and cells in contact; see, e.g., Ref. [7]) and (ii) an ellipse (expected for small forces) that touches each plate at one point.

The energy of the stadium profile is calculated as follows. Flat pieces of the profile contribute nothing to the elastic energy since there $K = 0$; the curved parts are two halves of a cylinder of height $h$ and radius $b$, where $K = b^{-1}$, so that we have

$$E = 2m_e gb + \frac{\pi \kappa h}{b}. \quad (3)$$

The spring will be in equilibrium when $dE / db = 0$, i.e., when

$$b = \sqrt{\frac{\kappa \pi h}{2m_e g}}. \quad (4)$$

This solution is expected to be correct only for sufficiently large loads. Note that the stadium profile fulfills the inextensibility requirement when $b < b_0$.

The elastic energy when $C$ is an ellipse with a circumference equal to $2\pi b_0$ (this is the inextensibility requirement; $b_0$ is the radius of the cylinder in its unladen state) can be expressed in terms of elliptic integrals. However, since this Ansatz makes sense only for small deformations where the major and minor axes of the ellipse, $a$ and $b$, are close, these integrals can be Taylor expanded to yield

$$\lim_{a \to b} E_{el} = \frac{\pi h \kappa}{b} \left( \frac{5(b_0/b) - 4(b_0/b)^2 - 2}{3 - 4(b_0/b)} \right). \quad (5)$$

The energy [Eq. (5)] differs from the elastic energy of the stadium profile by the multiplicative factor in large parentheses, which is smaller than one in the interval $b \in [0.80b_0, b_0]$, where the elliptic profile is the better Ansatz. From Eq. (5) we derive the spring equilibrium for small deformations: $b = b_0 - m_e gb_0^3 / 7\pi h \kappa$ (only first-order terms are kept).

The curve $C$ that minimizes the elastic energy for a given separation between the press plates can be found numerically. To this end we discretize the profile of the deformed spring in $N$ points and reformulate the elastic energy functional to depend on coordinates of these points. The functional is minimized using a particular variant of the conjugate gradient minimization (see, e.g., Refs. [3,11,14]). The constraints of the inextensibility of the sheet and the impenetrability of the top and bottom press surfaces are implemented through energy penalty for all configurations that violate the constraints.

In Fig. 2 we show the theoretical predictions for the spring energy. The circles show the numerical results, the dotted line is the prediction of the variational method based on the stadium profile, and the dashed line corresponds to the elliptic profile. The solid line shows the stadium profile energy multiplied by 0.912.

FIG. 2. (Color online) Theoretical predictions for the spring energies. The circles show the numerical results, the dotted line is the prediction of the variational method based on the stadium profile, and the dashed line corresponds to the elliptic profile. The solid line shows the stadium profile energy multiplied by 0.912.
appropriate scale of elastic energy is $\pi \kappa h/b_0$. The energy-
shape dependence can thus be written as
\[ \frac{b_0}{\pi \kappa h} E_{el} = \mathcal{U} \left( \frac{b}{b_0} \right) \equiv \bar{E}_{el}, \] (6)
where $\mathcal{U}(b/b_0)$ is the universal function characteristic for our
problem. The appropriately scaled energy (adimensional) is
donoted by an overline ($\bar{E}_{el}$), as all the adimensional quantities
will be in the following.

The tubes that were used in our experiments are constructed from
thin transparent foils (made of polymer material), which
are usually used for plastic covers for strip and spiral book
binding. Their size is that of A4 paper, which
will be in the following.

The tubes are made by rolling a foil in a cylinder, either
along its width or its length $L$ ($L > W$), and by using the
adhesive tape to fix the cylinder. For accurate measurements,
the width of the overlapping region where the adhesive tape
is applied should be as small as possible ($\sim 2$ mm in our
measurements). The response of the spring is measured in
a press with two parallel plates as illustrated in Fig. 1. The
upper plate is driven by a wing nut with a known pitch that
enables one to precisely determine the shift of the top plate. An
ordinary kitchen scale located below the lower plate measures the
force that the tube exerts on the plates.

In Fig. 3 we show four sets of measurements on four tubes
showing the half of the separation between the two press
surfaces, $b$, as a function of the mass read on the scale, $m_e$.
Every tube was made from nominally identical foils (denoted
set 1) from the same package with the thickness $190 \pm 7 \mu m$. We have rolled the foils along their longer side so
that $h = 210$ mm. We see that the dependence $b \propto m_e^{-1/2}$ is
obeyed by the data for sufficiently pressed foils, $b < 0.7(b_0)$.
From the numerical analysis we conclude that an easy, yet very
accurate way to obtain the bending rigidity of the foils is to fit
the experimental data to the
\[ b = \sqrt[4]{\frac{0.912 \pi h}{2 m_e g}} \] (7)
dependence in the region $b < 0.7 b_0$.

In addition to this, we investigate experimentally the
predicted universality of the system by studying different
springs. To this end, we have constructed two tubes from
set 1 foils by rolling them along their length and width. We
have tested two additional sets of sheets. For set 2 foils we
used binding covers (A4 format) of smaller thickness (146 \pm
8 $\mu m$). For set 3 we used A4 foils of thickness 412 $\pm 4 \mu m$.
For all measurements, we scaled the mass readings to produce
the adimensional experimental force $\bar{F}_{exp}$ as
\[ \bar{F}_{exp} = \frac{2 m_e g b_0^2}{\pi \kappa h}. \] (8)

The scale of force can be derived from the scale of energy
$\pi \kappa h/b_0$ simply by dividing it by the scale of length, $b_0$ in
our case. The quantity in Eq. (8) can be compared directly to its
counterpart obtained from the numerical analysis,
\[ \bar{F}_{num} = - \frac{d \bar{E}_{el}}{d(b/b_0)}. \] (9)

Note that the factor of 2 in Eq. (8) arises from the fact that
deformation of the tube where $b$ changes by $\Delta b$ requires
applying the force of $m_e g$ on a distance of $2 \Delta b$; the same
factor of 2 is present in Eq. (2). The comparison of the scaled
experimental readings with the numerical results is shown in
Fig. 4. One can see that the scaling predicted by the numerical
results is evident in the experimental data through an interval
of almost four orders of magnitude of the force (in both the small- and large-deformation regimes). In the inset of Fig. 4 we show the profiles obtained by the numerical method (due to symmetry, it is sufficient to show only quarters of the profiles) corresponding to different parts of the universal curve.

A summary of the bending rigidities obtained for the sheets shown in Fig. 4 is shown in Table I. The fifth column of data contains the bulk Young modulus $E$ of the sheets obtained from Eq. (1) using a value of $\kappa$ determined experimentally and a Poisson ratio of $\nu = 0.3$ that is typical for most materials. The bulk Young moduli are indeed in the range expected for polymeric materials such as nylon ($E \sim 2–4 \text{ GPa}$).

In conclusion, we have investigated theoretically and experimentally small and large radial deformations of a tube made from a sheet of thin elastic material. We have obtained a (scaled) universal response curve of the system. The numerical solution of the problem was compared with two simple Ansatz trial functions that represent the linear (ellipse) and nonlinear (stadium) responses of the tube. The stadium profile of the pressed cylinder is found to be an excellent approximation when compared to our numerical simulations and it predicts a correct analytical dependence of the energy and the reaction force of the deformed tube.

Note added in proof. We would like to mention a recently published paper related to our work by Kashcheyevs [18].

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